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GEOMETRICAL AND ANALYTICAL PROPERTIES OF SOME CLASSES OF UNIVALENT FUNCTIONS

Ph.D. Thesis Summary

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Cluj-Napoca
2011

Contents

Keywords: univalent function, function with positive real part, differential subordination, strong differential subordination, strong differential superordination, integral operator, Sălăgean differential operator, Ruscheweyh differential operator, convexity, uniform convexity, almost convexity of order α, harmonic function, multivalent harmonic function, differential inequality.

Introduction

The geometric theory of complex variable functions was set as a separately branch of complex analysis in the XX-th century when appeared the first important papers in this domain, owed to P. Koebe [57], I.W. Alexander [8], L. Bieberbach [14].

The univalent function notion occupies a central role in geometric theory of analytic functions, first paper dating since 1907 owed to P. Koebe. The study of univalent functions was continued by Plemelj, Gronwall and Faber.

Now exist many treated and monographs dedicated to univalent functions study. Among them we recall those of P. Montel, Z. Nehari, L.V. Ahlfors [3], Ch. Pommerenke [108], A.W. Goodman [40], P.L. Duren [32], D.J. Hallenbeck, T.H. MacGregor [49], S.S. Miller, P.T. Mocanu [76] and P.T. Mocanu, T. Bulboacă, Gr. Şt. Sălăgean [85].

The Mathematical Romanian School brought her valuable contribution in the geometric theory of univalent functions. Among them we mention two personalities from Cluj, namely G. Călugăreanu and P. T. Mocanu. G. C alugăreanu, the creator of the romanian school's theory of univalent functions, was the first mathematician who obtain in 1931 the necessary and sufficient conditions for univalence in the open unit disc. The researches initiated by G. Calugăreanu are continued by P. T. Mocanu, who obtained important results in the geometric theory of univalent functions: introducing α-convexity, getting univalence criteria for nonanalytic functions, development in collaboration with S. S. Miller the method of differential subordinations, and most recently the theory of differential superordinations. The method of differential subordinations has an important role in a much easier demonstration of already known results, as well as in many other new obtained results.

This paper presents a Ph.D. Thesis Summary. This includes the study of certain geometric properties, expressed analytically, of some classes of analytic complex variable functions.

Ph.D. Thesis is divided into 6 chapters and a bibliography containing 136 references, 16 of them belong to the author, 7 of those are written in collaboration.

In the following, in each chapter I selected the most relevant results, with the emphasis on my original contributions. The results from the first chapters, respectively the last chapter, sections 1-4, are renumbered. Finally, full bibliography is included.

Chapter 1

Definitions and classical results

The first chapter is divided into 20 sections and contains basic notions and results of geometric theory of functions, which will be used in the next chapters.

Classic results are enumerated, like: Riemann's Theorem, Area theorem, covering and deformation theorems for the S class of univalent and normal functions in the open unit disc, Bieberbach's conjecture, and some classes of univalent functions that are characterized by remarkable geometric properties expressed analytically by differentiable inequalities, namely the classes of starlike and convex functions, the classes of uniformly starlike, uniformly convex and α-convex functions.

The last 8 sections are dedicated to the differential subordination theory, known under the name "the method of admissible functions" and developed by S. S. Miller and P. T. Mocanu. Recently they have introduced the concept of "differential superordination", as a dual notion of the differential subordination. The strong differential subordination, respectively strong differential superordination are new concepts that come in addition.

The results of this chapter are contained mainly in the following papers: "Mathematical Analysis (Complex functions)", P. Hamburg, P. T. Mocanu, N. Negoescu, [51], "The geometric theory of univalent functions", P. T. Mocanu, T. Bulboacă, Gr. Şt. Sălăgean, [85], "Special chapters of Complex Analysis", G. Kohr, P. T. Mocanu, [59], as well as papers by S. S. Miller and P. T. Mocanu.

1.1 Univalent functions

Definition 1.1.1 [51] An holomorphic and injective function onto a domain D from $\mathbb{C}$ is called univalent on D.

We note with $\mathcal{H}_u(D)$ the set of univalent functions on D and with $\mathcal{H}(D)$ the set of holomorphic functions on the domain D.

Theorem 1.1.1 [51] *If $f \in \mathcal{H}_u(D)$ then $f'(z) \neq 0$ for all $z \in D$.*

Corollary 1.1.1 (Teorema lui Alexander) [117] *If D is a convex domain and $f \in \mathcal{H}(D)$ a.i.* Re $f'(z) > 0$, *for any $z \in D$, then $f \in \mathcal{H}_u(D)$.*

1.2 Conformal mappings

Definition 1.2.1 [51] Let be the domain D and Δ from $\mathbb{C}$, a function $f \in \mathcal{H}_u(D)$ so that $f(D) = \Delta$ is called conformal mapping of the domain D on the domain Δ. D and Δ are calling conform equivalent if there exists a conformal mapping of D on Δ.

Theorem 1.2.1 (Riemann) [51] *Every simply connected domain D, which is a proper subset of $\mathbb{C}$, can be mapped conformally onto the unit disc.*

1.3 The class S . Properties.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we note with

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : \ f(z) = a + a_n z^n + \dots\},$$
$$\mathcal{H}_n = \{f \in \mathcal{H}(U) : \ f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1} z^{n+1} + \dots\},$$

$\mathcal{A}_1 = \mathcal{A}$.
The class S is:

$$S = \{f \in \mathcal{H}_u(U) : \ f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad f(0) = f'(0) - 1 = 0, \quad z \in U\}.$$

Koebe's function

$$K_\theta(z) = \frac{z}{(1 + e^{i\theta} z)^2} \tag{1.3.1}$$

has an extremal role in class S .

Theorem 1.3.1 [85] *The class S is compact.*

1.4 The class Σ . Properties.

Throughout the present thesis we will use the following notation $U^- = \{z \in \mathbb{C}_\infty : \ |z| > 1\}$ for the external of the open unit disc.

We note with Σ the class of meromorphic functions φ with the unic pol $\zeta = \infty$ and univalent in the external of the open unit disc, which allowed the Laurent serie representation at ∞ of the form

$$\varphi(\zeta) = \zeta + \sum_{k=0}^{\infty} \frac{b_k}{\zeta^k}, \quad 1 < |z| < \infty.$$

The function from Σ are normalized with the conditions $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$.

We note with

$$\Sigma_0 = \{\varphi \in \Sigma : \ \varphi(\zeta) \neq 0, \ \zeta \in U^-\}.$$

Proposition 1.4.1 [85] *Between the classes S and Σ_0 exists a bijective, such as the class Σ is "more general" then class S.*

1.5 Area theorem.
Bieberbach's conjecture - De Branges Theorem

Theorem 1.5.1 (Gronwall) [47] *If consider*

$$\varphi(\zeta) = \zeta + \sum_{n=0}^{\infty} \frac{\alpha_n}{\zeta^n}$$

a function from the class Σ *, then*

$$aria\, E(\varphi) = \pi \left(1 - \sum_{n=1}^{\infty} n|\alpha_n|^2\right) \geq 0$$

and

$$\sum_{n=1}^{\infty} n|\alpha_n|^2 \leq 1$$

Theorem 1.5.2 (Bieberbach's conjecture - De Branges Theorem) [85] *If the function* $f(z) = z + a_2 z^2 + \ldots$ *belongs to the class* S*, then* $|a_n| \leq n$*,* $n = 2, 3, \ldots$ *with eqality for Koebe's function (1.3.1).*

1.6 Analytic functions with positive real part

Definition 1.6.1 [85]

1. By Charatheodory's functions class we understand

$$\mathcal{P} = \{p \in \mathcal{H}(U) : \; p(0) = 1, \; \text{Re}\, p(z) > 0, \; z \in U\}.$$

2. By Schwarz's functions class we understand:

$$\mathcal{B} = \{\varphi \in \mathcal{H}(U) : \; \varphi(0) = 0, \; |\varphi(z)| < 1, \; z \in U\}.$$

Over the next five sections, we remind some well-known classes of univalent functions class which afre characterized by remarkable geometric properties, analytically expressed by differentiable inequalities, namely the classes of starlike and convex functions, the classes of uniformly starlike, uniformly convex and α-convex functions.

1.7 Starlike functions

Definition 1.7.1 [85] We denote with S^* the class of functions $f \in \mathcal{A}$ which are starlike in the unit disk:

$$S^* = \left\{ f \in \mathcal{A} : \text{Re}\, \frac{zf'(z)}{f(z)} > 0, \; z \in U \right\}$$

Remark 1.7.1 S^* class is compact.

Definition 1.7.2 For $0 \leq \alpha < 1$, we define the set

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha,\ z \in U \right\} \tag{1.7.1}$$

called the class of starlike functions of order α.

Definition 1.7.3 For $0 < \alpha \leq 1$, we define

$$S^*[\alpha] = \left\{ f \in \mathcal{A} :\ \left| \arg \frac{zf'(z)}{f(z)} \right| < \alpha \frac{\pi}{2},\ z \in U \right\}$$

called the class of strong starlike functions of order α

1.8 Uniformly starlike functions

Definition 1.8.1 [42] The class of uniformly starlike functions is

$$US^* = \left\{ f \in S : \operatorname{Re} \left[\frac{f(z) - f(\xi)}{(z - \xi) f'(z)} \right] > 0,\ (z, \xi) \in U \times U, \right\}$$

Definition 1.8.2 [2] The class of uniformly starlike functions of order α is

$$US^*(\alpha) = \left\{ f \in S : \operatorname{Re} \left[\frac{f(z) - f(\xi)}{(z - \xi) f'(z)} \right] \geq \alpha, \quad (z, \xi) \in U \times U, \alpha \in [0, 1) \right\}$$

Observăm că $US^*(0) = US^*$.

1.9 Convex functions

Definition 1.9.1 [85] The set

$$K \overset{not}{=} S^c = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U \right\} \tag{1.9.1}$$

is called the class of convex functions, normalized with the conditions $f(0) = f'(0) - 1 = 0$.

Remark 1.9.1 We have $K \subset S^* \subset S$ and

$$f \in K \ \Leftrightarrow\ zf'(z) \in S^*.$$

Definition 1.9.2 The class of convex functions of order α is

$$K(\alpha) = \left\{ f \in A : \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > \alpha, \quad 0 \leq \alpha < 1 \right\}.$$

1.10 Uniformly convex functions

Definition 1.10.1 [43] The class of uniformly convex functions is

$$UCV = \left\{ f \in S : \text{Re}\,\{1 + \frac{f''(z)}{f'(z)}(z-\xi)\} \geq 0, ,\ (z,\xi) \in U \times U \right\}.$$

Definition 1.10.2 [113]

$$f(z) \in SP \Leftrightarrow \text{Re}\,\{\frac{zf'(z)}{f(z)}\} \geq |\frac{zf'(z)}{f(z)} - 1|,\ z \in U, \tag{1.10.1}$$

where SP is the class of uniformly starlike functions relative to the class UCV.

Remark 1.10.1

$$f(z) \in UCV \Leftrightarrow zf'(z) \in SP.$$

Definition 1.10.3 [113] A function $f \in S$ is in the class $SP(\alpha)$, if it is satisfied the analytical characterization:

$$\text{Re}\,\{\frac{zf'(z)}{f(z)} - \alpha\} \geq |\frac{zf'(z)}{f(z)} - 1|,\ \alpha \in \mathbb{R},\ z \in U, \tag{1.10.2}$$

and
$f(z) \in UCV(\alpha)$, the class of uniformly convex functions of order α, if and only if $zf'(z) \in SP(\alpha)$.

1.11 α-convex functions

Definition 1.11.1 [85] Let the function $f \in \mathcal{A}$ and $\alpha \in \mathbb{R}$. The function f is called α-convex in the open unit disc, if Re $J(\alpha, f; z) > 0$, $z \in U$, where

$$J(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right). \tag{1.11.1}$$

We note with M_α the class of α-convex functions.

1.12 Sălăgean, Ruscheweyh and Bernardi-Libera operators

Definition 1.12.1 [118] For $f \in \mathcal{H}(U)$, $n \in \mathbb{N}$, the operator I^n defined with:

$$\begin{aligned} &I^0 f(z) = f(z) \\ &I^1 f(z) = If(z) = zf'(z) \\ &\ldots \\ &I^n f(z) = I(I^{n-1}f(z)) = z[I^{n-1}f(z)]',\ z \in U,\ n > 1, \end{aligned}$$

is called Sălăgean differential operator.

Definition 1.12.2 For $f(z) \in \mathcal{H}(U)$, $n \in \mathbb{N}$, the operator R^n defined with:

$$\begin{aligned}
&R^0 f(z) = f(z) \\
&R^1 f(z) = zf'(z) \\
&2R^2 f(z) = z[R^1 f(z)]' + R^1 f(z) \\
&\dots \\
&(n+1)R^{n+1} f(z) = z[R^n f(z)]' + n[R^n f(z)], \quad z \in U,
\end{aligned}$$

is called Ruscheweyh differential operator.

Definition 1.12.3

$$L_a[f](z) = F(z) = \frac{1+a}{z^a} \int_0^z f(t) t^{a-1} dt,$$

is called Bernardi-Libera operator.

1.13 Subordinations principle. The method of admissible functions

Definition 1.13.1 [85] Let $f, g \in \mathcal{H}(U)$. We say that the function f is subordinate to the function g and note

$$f \prec g \quad \text{sau} \quad f(z) \prec g(z),$$

if there exists a function $w \in \mathcal{H}(U)$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ $(w \in \mathcal{B})$ such as

$$f(z) = g[w(z)], \quad z \in U.$$

Let $\Omega, \Delta \subset \mathbb{C}$, the function $p \in \mathcal{H}(U)$ with the property $p(0) = a$, $a \in \mathbb{C}$ and the function $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$.

Definition 1.13.2 [85] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and the function $h \in \mathcal{H}_u(U)$. If $p \in \mathcal{H}[a, n]$ satisfies the differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad z \in U \tag{1.13.1}$$

then p is called (a, n)-solution of this subordination.

The subordination (1.13.1) is called second order differential subordination and the function q univalent in U, is called (a, n)-dominant of the differential subordination, if $p(z) \prec q(z)$ for all function p which satisfies (1.13.1).

A dominant $\widetilde{q}$ which $\widetilde{q} \prec q$ for any q for the subordination (1.13.1) is called the best (a, n)-dominant.

Definition 1.13.3 [85] WE note with Q the set of functions q that are holomorphic and injective on the set $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Definition 1.13.4 [71], [73] Let $\Omega \subset \mathbb{C}$, $q \in Q$ şi $n \in \mathbb{N}^*$. The class of admissible functions is $\Psi_n[\Omega, q]$ a funcţiilor $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, and consists of those functions which satisfies the admissibility condition

$$\psi(r, s, t; z) \notin \Omega$$

whenever

$$r = q(\zeta), \quad s = m\zeta q'(\zeta), \quad \text{Re}\left[\frac{t}{s} + 1\right] \geq m\text{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right],$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$ şi $m \geq n$.

If $n = 1$, we note $\Psi_1[\Omega, q] = \Psi[\Omega, q]$.

1.14 First order linear differential subordinations

Definition 1.14.1 [72] A differential subordination

$$A(z)zp'(z) + B(z)p(z) \prec h(z) \tag{1.14.1}$$

or

$$zp'(z) + P(z)p(z) \prec h(z) \tag{1.14.2}$$

is call first order linear differential subordination.

1.15 Second order linear differential subordinations

Definition 1.15.1 [72] A differential subordination

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1.15.1}$$

where A, B, C, D şi h are complex functions, or more general

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega \tag{1.15.2}$$

were $\Omega \subset \mathbb{C}$, is call second order linear differential subordination.

1.16 Briot-Bouquet differential subordinations

Definition 1.16.1 [76] Let β and $\gamma \in \mathbb{C}$, $\beta \neq 0$, $h \in \mathcal{H}_u(U)$ with $h(0) = a$ and let $p \in \mathcal{H}[a, n]$ which verify the relation:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{1.16.1}$$

This differential subordination is call Briot-Bouquet differential subordinations.

1.17 Strong differential subordinations

Let $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Definition 1.17.1 [9] Let $h(z,\zeta)$ an analytic function in $U \times \overline{U}$ and let $f(z)$ an analytic and univalent function in U. The function $f(z)$ is said to be strongly subordinate to $h(z,\zeta)$,or $h(z,\zeta)$ is said to be strongly superordinate to $f(z)$, and we write $f(z) \prec\prec h(z,\zeta)$, if $f(z)$ is subordinate to the function $h(z,\zeta)$, after z, for all $\zeta \in \overline{U}$.
If $h(z,\zeta)$ is an univalent function in U, for all $\zeta \in \overline{U}$, then $f(z) \prec\prec h(z,\zeta)$ if $f(0) = h(0,\zeta)$ and $f(U) \subset h(U \times \overline{U})$.

Remark 1.17.1 If $h(z,\zeta) \equiv h(z)$ then strong superordination becomes usual notion of superordination.

Next we define the following classes in $U \times \overline{U}$:

$$\mathcal{H}\zeta[a,n] = \{f \in \mathcal{H}(U \times \overline{U}) :\ f(z,\zeta) = a + a_n(\zeta)z^n + a_{n+1}(\zeta)z^{n+1} + \cdots\}$$

with $z \in U$, $\zeta \in \overline{U}$, $a_k(\zeta)$ holomorphic functions in $\overline{U}$, $k \geq n$,

$$\mathcal{H}\zeta_u(U) = \{f \in \mathcal{H}\zeta[a,n] :\ f(\cdot,\zeta) \text{ univalent in U, pentru toţi } \zeta \in \overline{U}\},$$

$$\mathcal{A}\zeta_n = \{f \in \mathcal{H}\zeta[a,n] :\ f(z,\zeta) = z + a_2(\zeta)z^2 + \cdots + a_n(\zeta)z^n + \cdots,\ z \in U, \zeta \in \overline{U}\}$$

with $\mathcal{A}\zeta_1 = \mathcal{A}\zeta$, and

$$\mathcal{S}\zeta = \{f \in \mathcal{A}\zeta_n :\ f(z,\zeta) \text{ univalent in } U \times \overline{U},\ z \in U, \text{ pentru toţi } \zeta \in \overline{U}\}.$$

Let

$$S^*\zeta = \left\{f \in \mathcal{A}\zeta :\ \operatorname{Re}\frac{zf'(z,\zeta)}{f(z,\zeta)} > 0,\ z \in U, \text{ pentru toţi } \zeta \in \overline{U}\right\}$$

the class of starlike functions in $U \times \overline{U}$,

$$K\zeta = \left\{f \in \mathcal{A}\zeta :\ \operatorname{Re}\frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0,\ z \in U, \text{ pentru toţi } \zeta \in \overline{U}\right\}$$

the class of convex functions in $U \times \overline{U}$,

$$C\zeta = \left\{f \in \mathcal{A}\zeta :\ \exists\varphi \in K\zeta, \operatorname{Re}\frac{f'(z,\zeta)}{\varphi'(z,\zeta)} > 0,\ z \in U, \text{ pentru toţi } \zeta \in \overline{U}\right\}$$

the class of close-to-convex functions in $U \times \overline{U}$.

1.18 Differential superordinations

The dual problem of differential subordinations, that of subordinations determination for differential superordinations was initiated in 2003 by S.S. Miller and P.T. Mocanu [77].

Theorem 1.18.1 [77] *Let* $q \in \mathcal{H}[a,n]$, *h analytic and* $\varphi \in \Phi_n[h,q]$. *If* $p \in Q(a)$ *and* $\varphi(p(z), zp'(z), z^2p''(z); z)$ *is univalent in* U, *then*

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.18.1}$$

implies $q(z) \prec p(z)$.

Theorem 1.18.2 [77] *Let h analytic in* U *and* $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. *We assume that the differential subordinatian*

$$\varphi(q(z), zq'(z), z^2q''(z); z) = h(z) \tag{1.18.2}$$

has a solution $q \in Q(a)$. *If* $\varphi \in \Phi[h,q]$, $p \in Q(a)$ *and* $\varphi(p(z), zp'(z), z^2p''(z); z)$ *is univalent in* U, *then*

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.18.3}$$

implies $q(z) \prec p(z)$ *and* q *is the best subordinant.*

1.19 Briot-Bouquet differential superordinations

If the sets $\Omega_1, \Omega_2, \Delta_1, \Delta_2 \subset \mathbb{C}$ simply connected domain, we can have:

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \;\Rightarrow\; p(z) \prec q_2(z) \tag{1.19.1}$$

and

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \;\Rightarrow\; q_1(z) \prec p(z). \tag{1.19.2}$$

We call Briot-Bouquet differential superordinations the left part of the relation (1.19.2), and the function q_1 is called a subordinant of the Briot-Bouquet differential superordinations.

1.20 Strong differential superordinations

The dual concept of strong differential superordination was introduced and developed by G. I. Oros and Gh. Oros [90], [95].

Let Ω a set from the complex plain $\mathbb{C}$, let p an analytic function in U and let $\psi(r,s,t;z,\zeta) : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$.

Definition 1.20.1 [92] We note with Q the set of functions $q(\cdot,\zeta)$ analytic and injective in z, for all $\zeta \in \overline{U}$, defined on $\overline{U} - E(q)$, where

$$E(q) = \{\xi \in \partial U : \lim_{z\to\xi} q(z,\zeta) = \infty, \quad z \in U, \zeta \in \overline{U}\}.$$

The subclass of Q for that $f(0,\zeta) \equiv a$ is noted with $Q(a)$.

Definition 1.20.2 [93] Let Ω_ζ be a set in $\mathbb{C}$, $q(\cdot,\zeta) \in \Omega_\zeta$ and n be a positive integer. The class of admissible functions $\psi_n[\Omega_\zeta, q(\cdot,\zeta)]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$(A) \qquad\qquad \psi(r, s, t; z, \zeta) \notin \Omega_\zeta,$$

whenever $r = q(\zeta,\zeta),\ s = m \cdot \zeta \cdot q'(\zeta,\zeta)$, $\operatorname{Re}\left[\frac{t}{s} + 1\right] \geq m\operatorname{Re}\left[\frac{\zeta q''(\zeta,\zeta)}{q'(\zeta,\zeta)} + 1\right]$, $z \in U$, $\zeta \in \partial U \setminus E(q)$, $\zeta \in \overline{U}$ and $m \geq n$. We write $\psi_1[\Omega_\zeta, q(\cdot,\zeta)]$ as $\psi[\Omega_\zeta, q(\cdot,\zeta)]$.

Definition 1.20.3 [95] Let $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ and let $h(z,\zeta)$ be analytic in $U \times \overline{U}$. If $p(z,\zeta)$ and $\varphi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta)$ are univalent in U for all $\zeta \in \overline{U}$ and satisfy the (second-order) strong differential superordination

$$(1.20.1) \qquad\qquad h(z,\zeta) \prec\prec \varphi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta),$$

then $p(z,\zeta)$ is called a solution of the strong differential superordination. An analytic function $q(z,\zeta)$ is called a subordinant of the solutions of the strong differential superordination, or more simply a subordinant if $q(z,\zeta) \prec\prec p(z,\zeta)$ f or all $p(z,\zeta)$ satisfying (1.20.1). An univalent subordinant $\widetilde{q}(z,\zeta)$ that satisfies $q(z,\zeta) \prec\prec \widetilde{q}(z,\zeta)$ for all subordinants $q(z,\zeta)$ of (1.20.1), is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

Chapter 2

Differential inequalities for univalent functions

The second chapter is devoted entirely to the differential inequalities.

2.1 Differential inequalities for functions with positive real part

In this section we intended to present certain conditions for the complex-valued functions $A, B : U \to \mathbb{C}$ defined in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ such that the differential inequality

$$\text{Re } [A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 - 3a\beta\left(zp'(z) - \frac{b}{2}\right)^2 +$$

$$+3a^2\gamma(zp'(z)) + \delta] > 0$$

implies Re $p(z) > 0$, where $p \in \mathcal{H}[1, n]$, $a \geq 0$, $\alpha, \beta, \gamma \in \mathbb{C}$. The above inequality is a generalization of a certain inequality obtained earlier by B. A. Frasin [34]. Some related results are also provide.

The results of this section are original, and are contained in the paper [125]. Particularizing the coefficients values the results from the papers [126] and [127] are highlighted.

Following the work done in A. Cătaş [23] we obtain the next theorem.

Theorem 2.1.1 [125] *Let* $a, b \in \mathbb{R}_+$, $\alpha, \beta, \gamma \in \mathbb{C}$, Re $\alpha \geq 0$, $\alpha + \beta \in \mathbb{R}_+$, $\alpha a + \beta b + \gamma a \in \mathbb{R}_+$,

$$\delta < \left(\frac{n^3}{8} + a^3\right) \text{Re } \alpha + \frac{3an^2}{4}(\alpha + \beta) + \frac{3an}{2}(\alpha a + \beta b + \gamma a) + \frac{3ab^2}{4}\text{Re } \beta$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i)\text{Re } A(z) > -\frac{3n^3}{8}\text{Re } \alpha - \frac{3an^2}{2}(\alpha + \beta) - \frac{3an}{2}(\alpha a + \beta b + \gamma a);$$

$$(ii)\text{Im}^2 B(z) \leq 4\left[\frac{3n^3}{8}\text{Re } \alpha + \frac{3an^2}{2}(\alpha + \beta) + \frac{3an}{2}(\alpha a + \beta b + \gamma a) + \text{Re } A(z)\right] \cdot \tag{2.1.1}$$

$$\cdot\left[\left(\frac{n^3}{8} + a^3\right)\text{Re } \alpha + \frac{3an^2}{4}(\alpha + \beta) + \frac{3an}{2}(\alpha a + \beta b + \gamma a) + \frac{3ab^2}{4}\text{Re } \beta - \delta\right].$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re}\left[A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 - 3a\beta\left(zp'(z) - \frac{b}{2}\right)^2 + \right. \tag{2.1.2}$$

$$\left. +3a^2\gamma(zp'(z)) + \delta\right] > 0$$

then

$$\text{Re } p(z) > 0.$$

Remark 2.1.1 For $a = 1$ similar results were obtained in [23]. For $b = 0$ the author obtained results in [126], and for $a = 1$, $b = 0$ and $\delta = 1$ the author has results in [127].

Taking $\beta = \gamma = \bar{\alpha}$ in the Theorem 2.1.1, we have

Corollary 2.1.1 [125] *Let* $a, b \in \mathbb{R}_+$, $\alpha \in \mathbb{C}$, $\text{Re } \alpha \geq 0$,

$$\delta < \left(\frac{n^3}{8} + a^3 + \frac{3an^2}{2} + \frac{3an}{2}(2a+b) + \frac{3ab^2}{4}\right) \cdot \text{Re } \alpha$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &(i) \text{Re } A(z) > \left[-\frac{3n^3}{8} - 3an^2 - \frac{3an}{2}(2a+b)\right] \text{Re } \alpha; \\ &(ii) \text{Im}^2 B(z) \leq 4 \cdot \left[\left(\frac{3n^3}{8} + 3an^2 - \frac{3an}{2}(2a+b)\right) \cdot \text{Re } \alpha + \text{Re } A(z)\right] \cdot \\ &\cdot \left[\left(\frac{n^3}{8} + a^3 + \frac{3an^2}{2} + \frac{3an}{2}(2a+b) + \frac{3ab^2}{4}\right) \text{Re } \alpha - \delta\right]. \end{aligned} \tag{2.1.3}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re}\left[A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 - 3a\bar{\alpha}\left(zp'(z) - \frac{b}{2}\right)^2 + \right. \tag{2.1.4}$$

$$\left. +3a^2\bar{\alpha}(zp'(z)) + \delta\right] > 0$$

then

$$\text{Re } p(z) > 0.$$

Taking $\alpha + \beta = \alpha a + \beta b + \gamma a = \alpha + \gamma = 1$ in Theorem 2.1.1, we obtain

Corollary 2.1.2 [125]*Let* $a, b \in \mathbb{R}_+$, $\alpha \in \mathbb{C}$, $\text{Re } \alpha \geq 0$,

$$\delta < \left(\frac{n^3}{8} + a^3\right) \text{Re } \alpha + \frac{3an^2}{4} + \frac{3an}{2} + \frac{3ab^2}{4}(1-\alpha)$$

and n *be a positive integer. Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &(i) \text{ Re } A(z) > -\frac{3n^3}{8}\text{Re } \alpha - \frac{3an^2}{2} - \frac{3an}{2}; \\ &(ii) \text{ Im}^2 B(z) \leq 4 \cdot \left[\frac{3n^3}{8}\text{Re } \alpha + \frac{3an^2}{2} + \frac{3an}{2} + \text{Re } A(z)\right] \cdot \\ &\cdot \left[\left(\frac{n^3}{8} + a^3\right) \text{Re } \alpha + \frac{3an^2}{4} + \frac{3an}{2} + \frac{3ab^2}{4}(1-\alpha) - \delta\right]. \end{aligned} \tag{2.1.5}$$

If $p \in \mathcal{H}[1,n]$ *and*

$$\text{Re}\,\left[A(z)p^2(z) + B(z)p(z) + \alpha(zp'(z) - a)^3 - 3a(1-\alpha)\left(zp'(z) - \frac{b}{2}\right)^2 + \right. \tag{2.1.6}$$

$$\left. +3a^2(1-\alpha)(zp'(z)) + \delta\right] > 0$$

then

$$\text{Re}\; p(z) > 0.$$

2.2 Differential inequalities for univalent functions

In this section we use a parabolic region to prove certain inequalities for uniformly univalent functions in the open unit disk U. By applying Sălăgean differential operator of a holomorphic function, we obtain conditions for belonging to the following classes: the class of uniformly starlike functions of order α, the class of uniformly convex functions of order α, and the class of uniformly close-to-convex functions of order α. The results of this section are original, and are contained in the paper [128].

Next theorem offer a sufficient condition for uniform stelarity:

Theorem 2.2.1 [128] *Let* $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$. *If the differential operator* $I^n f$ *satisfies the following inequality:*

$$\text{Re}\,\left(\frac{\frac{I^{n+2}f(z)}{I^{n+1}f(z)} - 1}{\frac{I^{n+1}f(z)}{I^n f(z)} - 1}\right) < \frac{5}{3}, \tag{2.2.1}$$

then $I^n f(z)$ *is uniformly starlike in* U.

We introduce a sufficient coefficient bound for uniformly starlike functions in the following theorem:

Theorem 2.2.2 [128] *Let* $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$, *and the differential operator* $I^n f$. *If*

$$\sum_{k=2}^{\infty}(2k+1-\alpha)|a_{k+1}| < 1-\alpha$$

then $I^n f(z) \in SP(\alpha)$.

Theorem 2.2.3 [128] *Let* $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$. *If the differential operator* $I^n f$ *satisfies the following inequality:*

$$\text{Re}\,\left[\frac{\frac{I^{n+3}f(z) - I^{n+2}f(z)}{I^{n+2}f(z) - I^{n+1}f(z)} - 2}{\frac{I^{n+2}f(z)}{I^{n+1}f(z)} - 1}\right] < 3, \tag{2.2.2}$$

then $I^n f(z)$ *is uniformly convex in* U.

We determine the sufficient coefficient bound for uniformly convex functions in the next theorem:

Theorem 2.2.4 [128] *Let $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$, and the differential operator $I^n f$. If*

$$\sum_{k=2}^{\infty}(k+1)(2k+1-\alpha)|a_{k+1}| < 1-\alpha, \tag{2.2.3}$$

then $I^n f(z) \in UCV(\alpha)$.

The following theorems give the sufficient conditions for uniformly close-to-convex functions.

Theorem 2.2.5 [128] *Let $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$. If the differential operator $I^n f$ satisfies the following inequality:*

$$\operatorname{Re}\left(\frac{I^{n+2}f(z)}{I^{n+1}f(z)} - 1\right) < \frac{1}{3}, \tag{2.2.4}$$

then $I^n f(z)$ is uniformly close-to-convex in U.

Theorem 2.2.6 [128] *Let $f \in \mathcal{A}$, $n \in \mathbb{N}^* \cup \{0\}$, and the differential operator $I^n f$. If $I^n f(z)$ satisfies the following inequality:*

$$\sum_{k=2}^{\infty}(k+1)|a_{k+1}| < \frac{1-\alpha}{2}, \tag{2.2.5}$$

then $I^n f(z) \in UCC(\alpha)$.

2.3 Sufficient conditions for univalence of certain integral operators

In this section we determine sufficient conditions for univalence of some integral operators, using certain univalent criteria, obtained by Ahlfors [3], Becker [11] and Pascu [101]. The results of this section are obtained in collaboration, and are contained in the paper [134], respectively [135].

Theorem 2.3.1 [134] *Let $M \geq 1$ and α with $\operatorname{Re}\alpha > 0$ be a complex number, $\alpha \neq 1$, and c be a complex number, with $|c| \leq 1$, $c \neq -1$. Let the function $g \in A$, satisfies the conditions*

$$\left|\frac{g(z)}{z}\right| \leq 3M-2 \tag{2.3.1}$$

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| \leq \frac{1}{3M-2}, \tag{2.3.2}$$

for all $z \in U$, and

$$|c| + \frac{3|\alpha-1|}{|\alpha|} \leq 1, \tag{2.3.3}$$

then the function

$$G_{\alpha,M}(z) = \left[\frac{\alpha}{M}\int_0^z u^{\frac{\alpha}{M}-1}\left[\frac{g(u)}{u}\right]^{\frac{\alpha-1}{M^2}} du\right]^{\frac{M}{\alpha}} \tag{2.3.4}$$

is in the class S.

Remark 2.3.1 For $M = 1$, we obtain the result from V. Pescar [105].

Theorem 2.3.2 [134] *Let $M \geq 1$ and α with $\mathrm{Re}\ \alpha > 0$ be a complex number, $\alpha \neq 1$, and β be a complex number with $\mathrm{Re}\ \beta > \mathrm{Re}\ \alpha$. Let the function g satisfies the condition*

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < \frac{M}{3} \tag{2.3.5}$$

for all $z \in U$, and

$$|\alpha| < 3\mathrm{Re}\ \alpha, \tag{2.3.6}$$

then the function

$$F_{\alpha,\beta,M}(z) = \left[\beta \int_0^z u^{\beta-1} \left[\frac{g(u)}{u}\right]^{\frac{\alpha}{M}} du\right]^{\frac{1}{\beta}} \tag{2.3.7}$$

is in the class S.

Remark 2.3.2 For $M = 1$, $\beta = 1$, the result was obtained in [11].

Theorem 2.3.3 [135] *Let $M \geq 1$ and α with $\mathrm{Re}\ \alpha > 0$ be a complex number and c be a complex number, with $|c| \leq 1$, $c \neq -1$. Let the function g satisfies the condition*

$$|\frac{g(z)}{z}| \leq M \tag{2.3.8}$$

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| \leq \frac{2M-1}{M} \tag{2.3.9}$$

for all $z \in U$, and

$$|c| + 3|\alpha - 1| \leq 1 \tag{2.3.10}$$

then the function

$$F_{\alpha,M}(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{\frac{\alpha-1}{M}} du \tag{2.3.11}$$

is in the class S.

Remark 2.3.3 For $M = 1$, the condition(2.3.9) expresses a sufficient condition for univalence of function g and this result can be found in ([100], Lema C).

Chapter 3

Differential subordinations

The third chapter consists of a single section which contains original results, obtained in collaboration, that can be found in the paper [136].

3.1 Univalent functions defined by Sălăgean differential operator

By using a certain operator I^n, we introduce a class of holomorphic functions $S_n(\beta)$, and obtain some subordination results. We also show that the set $S_n(\beta)$ is convex and obtain some new differential subordinations related to certain integral operators.

Definition 3.1.1 [97] If $0 \leq \beta < 1$ and $n \in \mathbb{N}$, we let $S_n(\beta)$ stand for the class of functions $f \in \mathcal{A}$, which satisfy the inequality

$$\operatorname{Re}(S^n f)'(z) > \beta \quad (z \in U).$$

Theorem 3.1.1 [136] *The set $S_n(\beta)$ is convex.*

Theorem 3.1.2 [136] *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad z \in U,$$

where c is a complex number, with $\operatorname{Re} c > -2$.
If $f \in S_n(\beta)$ and $F = I_c(f)$ where

$$F(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t) dt, \quad \operatorname{Re} c > -2, \tag{3.1.1}$$

then

$$[I^n f(z)]' \prec h(z), \quad z \in U \tag{3.1.2}$$

implies

$$[I^n F(z)]' \prec q(z), \quad z \in U,$$

and this result is sharp.

Theorem 3.1.3 [136] *Let* Re $c > -2$ *and*

$$w = \frac{1 + |c+2|^2 - |c^2 + 4c + 3|}{4\text{Re }(c+2)}. \tag{3.1.3}$$

Let h *be an analytic function in* U *with* $h(0) = 1$ *and suppose that*

$$\text{Re } \frac{zh''(z)}{h'(z)} + 1 > -w.$$

If $f \in S_n(\beta)$ *and* $F = I_c(f)$, *where* F *is defined by (3.1.1), then*

$$[I^n f(z)]' \prec h(z), \quad z \in U \tag{3.1.4}$$

implies

$$[I^n F(z)]' \prec q(z), \quad z \in U,$$

where q *is the solution of the differential equation*

$$q(z) + \frac{1}{c+2} zq'(z) = h(z), \quad h(0) = 1,$$

given by

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t)dt, \quad z \in U.$$

Moreover q *is the best dominant.*

Remark 3.1.1 [136] If we put

$$h(z) = \frac{1 + (2\beta - 1)z}{1+z}$$

in Theorem 3.1.3, we obtain the following interesting result.

Corollary 3.1.1 [136] *If* $0 \le \beta < 1$, $n \in \mathbb{N}$, $\text{Re}\, c > -2$ *and* I_c *is defined by (3.1.1), then*

$$I_c[S_n(\beta)] \subset S_n(\delta),$$

where $\delta = \min_{|z|=1} \text{Re } q(z) = \delta(c, \beta)$, *and this results is sharp. Moreover*

$$\delta = \delta(c, \beta) = 2\beta - 1 + (c+2)(2 - 2\beta)\sigma(c), \tag{3.1.5}$$

where

$$\sigma(x) = \int_0^1 \frac{t^{x+1}}{1+t} dt. \tag{3.1.6}$$

Chapter 4

Strong differential subordinations

The fourth chapter is reserved to the strong differential subordinations, and is composed of three sections that contain only original results can be found in the papers [129], [130] and [131].

4.1 Strong differential subordinations obtained by the medium of an integral operator

In this section we define the class $S_n^m(\alpha)$, and we study strong differential superordinations obtained using the properties of Sălăgean integral operator. The results are original and can be found in the paper [129].

Definition 4.1.1 [129] Let $\alpha > 1$ and $m, n \in \mathbb{N}$. We denote by $S_n^m(\alpha)$ the set of functions $f \in A\zeta_n$ that satisfy the inequality

$$Re[\mathtt{I}^m f(z,\zeta)]'_z > \alpha, \quad z \in U, \zeta \in \overline{U}.$$

Theorem 4.1.1 [129] *If $\alpha < 1$, and $m, n \in \mathbb{N}$, then*

$$S_n^m(\alpha) \subset S_n^{m+1}(\delta),$$

where

$$\delta = \delta(\alpha, \zeta, n) = 2\alpha - \zeta + \frac{2(\zeta - \alpha)}{n}\sigma(\frac{1}{n})$$

and

$$\sigma(x) = \int_0^1 \frac{t^{x-1}}{1+t}dt. \tag{4.1.1}$$

Theorem 4.1.2 [129] *Let $h(z,\zeta)$ an analytic function from $U \times \overline{U}$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, $\zeta \in \overline{U}$, that satisfies inequality*

$$Re[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2}.$$

If $f(z,\zeta) \in A\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{4.1.2}$$

then

$$[I^{m+1}f(z,\zeta)]' \prec\prec g(z,\zeta),$$

where

$$g(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt.$$

The function g is convex and is the best dominant.

Theorem 4.1.3 [129] *Let $g(z,\zeta) \in K\zeta$ a function with $g(0,\zeta) = 1$ and suppose that*

$$h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta), \quad z \in U, \zeta \in \overline{U}.$$

If $f(z,\zeta) \in \mathcal{A}\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{4.1.3}$$

then

$$[I^{m+1}f(z,\zeta)]' \prec\prec g(z,\zeta).$$

Theorem 4.1.4 [129] *Let $g(z,\zeta) \in K\zeta$ a function with $g(0,\zeta) = 1$, and $h(z,\zeta)$ given by*

$$h(z,\zeta) = g(z,\zeta) + nzg'(z,\zeta).$$

If $f(z,\zeta) \in \mathcal{A}\zeta_n$ and verify the strong differential subordination

$$[I^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{4.1.4}$$

then

$$\frac{I^m f(z,\zeta)}{z} \prec\prec g(z,\zeta).$$

4.2 Strong differential subordination obtained by Ruscheweyh operator

In this section we define a class of univalent functions $\mathcal{R}\zeta_n^m(\alpha)$, and we study new strong differential subordinations using Ruscheweyh derivative. The results are original and can be found in the paper [130].

Definition 4.2.1 [130] Let $\alpha < 1$ and $m, n \in \mathbb{N}$. We denote by $\mathcal{R}\zeta_n^m(\alpha)$ the set of functions $f \in A\zeta_n$, that satisfies the inequality

$$Re[R^m f(z,\zeta)]'_z > \alpha, \quad z \in U, \zeta \in \overline{U}. \tag{4.2.1}$$

Theorem 4.2.1 [130] *If $\alpha < 1$, and $m, n \in \mathbb{N}$, then $\mathcal{R}\zeta_n^{m+1}(\alpha) \subset \mathcal{R}\zeta_n^m(\delta)$, where*

$$\delta = \delta(\alpha,\zeta,n,m) = 2\alpha - \zeta + 2(\zeta-\alpha)\frac{m+1}{n}\sigma(\frac{m+1}{n}) \tag{4.2.2}$$

and

$$\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t}dt. \tag{4.2.3}$$

Theorem 4.2.2 [130] *Let* $q(z,\zeta) \in K\zeta$ *be a function with* $q(0,\zeta) = 1$, *and let* $h(z,\zeta)$ *be an analytic function given by*

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} zq'(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \tag{4.2.4}$$

If $f \in A\zeta_n$ *and the strong differential subordination*

$$[R^{m+1}f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{4.2.5}$$

holds, then

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta)$$

and this is the best result.

Theorem 4.2.3 [130] *Let* $h(z,\zeta)$ *an analytic function from* $U \times \overline{U}$, *with* $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, *that satisfies the inequality*

$$Re[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2(m+1)}, m \geq 0. \tag{4.2.6}$$

If $f(z,\zeta) \in A\zeta_n$ *and the strong differential subordination*

$$[R^{m+1}f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{4.2.7}$$

holds, then

$$[R^m f(z,\zeta)]' \prec\prec q(z,\zeta),$$

where

$$q(z,\zeta) = \frac{m+1}{nz^{\frac{m+1}{n}}} \int_0^z h(t,\zeta) t^{\frac{m+1}{n}-1} dt. \tag{4.2.8}$$

The function $q(z,\zeta) \in K\zeta$ *and is the best dominant.*

Theorem 4.2.4 [130] *Let* $q(z,\zeta) \in K\zeta$ *be a function with* $q(0,\zeta) = 1$ *and suppose that* $h(z,\zeta) = q(z,\zeta) + nzq'(z,\zeta)$, $z \in U, \zeta \in \overline{U}, n \in \mathbb{N}$. *If* $f(z,\zeta) \in A\zeta_n$ *and the strong differential subordination*

$$[R^m f(z,\zeta)]' \prec\prec h(z,\zeta), \tag{4.2.9}$$

holds, then

$$\frac{[R^m f(z,\zeta)]}{z} \prec\prec q(z,\zeta).$$

4.3 Subclasses of $\alpha-$uniformly convex functions obtained by using an integral operator and the theory of strong differential subordinations

In this section we define some subclasses of $\alpha-$uniformly convex functions with respect to a convex domain included in the right half plane D, obtained by using an integral operator and the theory of strong differential subordinations.

The results are original and can be found in the paper [131].

Definition 4.3.1 [12] Let consider the integral operator $L_a : \mathcal{A}\zeta_n \to \mathcal{A}\zeta_n$ defined as:

$$f(z,\zeta) = L_a F(z,\zeta) = \frac{1+a}{z^a}\int_0^z F(t,\zeta)t^{a-1}dt, \quad a \in \mathbb{C}, \quad \mathrm{Re}\, a \geq 0. \tag{4.3.1}$$

In the case $L_a : \mathcal{A} \to \mathcal{A}$, $a = 1, 2, 3, \cdots$, this operator was introduced by S.D.Bernardi [12].

Definition 4.3.2 [67] Let $\alpha \in [0,1]$ and $f(z,\zeta) \in \mathcal{A}\zeta_n$. We say that f is a $\alpha-$uniformly convex function if

$$\mathrm{Re}\,\{(1-\alpha)\frac{zf'(z,\zeta)}{f(z,\zeta)} + \alpha(1 + \frac{zf''(z,\zeta)}{f'(z,\zeta)})\} \geq |(1-\alpha)(\frac{zf'(z,\zeta)}{f(z,\zeta)} - 1) + \alpha\frac{zf''(z,\zeta)}{f'(z,\zeta)}|, \quad z \in U, \zeta \in \overline{U}.$$

We denote this class by $UM\zeta_\alpha$.

Definition 4.3.3 [67] Let $\alpha \in [0,1]$ and $n \in \mathbb{N}$. We say that $f(z,\zeta) \in \mathcal{A}\zeta_n$ is in the class $UD\zeta_{n,\alpha}(\beta,\gamma), \beta \geq 0, \gamma \in [-1,1), \beta+\gamma \geq 0$ if

$$\mathrm{Re}\left[(1-\alpha)\frac{I^{n+1}f(z,\zeta)}{I^n f(z,\zeta)} + \alpha\frac{I^{n+2}f(z,\zeta)}{I^{n+1}f(z,\zeta)}\right] \geq$$

$$\geq \beta\left|(1-\alpha)\frac{I^{n+1}f(z,\zeta)}{I^n f(z,\zeta)} + \alpha\frac{I^{n+2}f(z,\zeta)}{I^{n+1}f(z,\zeta)} - 1\right| + \gamma.$$

Definition 4.3.4 [15] The function $f(z,\zeta) \in \mathcal{A}\zeta_n$ is n-starlike with respect to convex domain included in right half plane D if the differential expression $\frac{I^{n+1}f(z,\zeta)}{I^n f(z,\zeta)}$ takes values in the domain D.

Remark 4.3.1 If we consider $q(z,\zeta)$ an univalent function with $q(0,\zeta) = 1$, $\mathrm{Re}\, q(z,\zeta) > 0$, $q'(0,\zeta) > 0$ which maps the unit disc U into the convex domain D we have:

$$\frac{I^{n+1}f(z,\zeta)}{I^n f(z,\zeta)} \prec\prec q(z,\zeta).$$

We denote by $S^*\zeta_n(q)$ the class of all these functions.

Let $q(z,\zeta)$ be an univalent function with $q(0,\zeta) = 1$, $q'(0,\zeta) > 0$, which maps the unit disc U into a convex domain included in right half plane D.

Definition 4.3.5 [1] Let $f(z,\zeta) \in \mathcal{A}\zeta_n$ and $\alpha \in [0,1]$. We say that f is a $\alpha-$uniform convex function with respect to D, if

$$J(\alpha, f; z, \zeta) = (1-\alpha)\frac{zf'(z,\zeta)}{f(z,\zeta)} + \alpha(1 + \frac{zf''(z,\zeta)}{f'(z,\zeta)}) \prec\prec q(z,\zeta).$$

We denote this class by $UM\zeta_\alpha(q)$.

Theorem 4.3.1 [131] *For all $\alpha, \alpha' \in [0,1]$ with $\alpha < \alpha'$ we have $UM\zeta_{\alpha'}(q) \subset UM\zeta_\alpha(q)$.*

Theorem 4.3.2 [131] *If $F(z,\zeta) \in UM\zeta_\alpha(q)$ then $f(z,\zeta) = L_a F(z,\zeta) \in S^*\zeta_0(q)$, where L_a is the integral operator defined by (4.3.1) and $\alpha \in [0,1]$.*

Definition 4.3.6 [1] Let $f(z,\zeta) \in \mathcal{A}\zeta_n$ and $\alpha \in [0,1]$, $n \in \mathbb{N}$. We say that f is an $\alpha - n-$uniformly convex function with respect to D, if

$$J_n(\alpha, f; z, \zeta) = (1-\alpha)\frac{I^{n+1}f(z,\zeta)}{I^n f(z,\zeta)} + \alpha\frac{I^{n+2}f(z,\zeta)}{I^{n+1}f(z,\zeta)} \prec\prec q(z,\zeta).$$

We denote this class by $UD\zeta_{n,\alpha}(q)$.

Theorem 4.3.3 [131] *For all $\alpha, \alpha' \in [0,1]$ with $\alpha < \alpha'$ we have $UD\zeta_{n,\alpha'}(q) \subset UD\zeta_{n,\alpha}(q)$.*

Theorem 4.3.4 [131] *If $F(z,\zeta) \in UD\zeta_{n,\alpha}(q)$, then $f(z,\zeta) = L_a F(z,\zeta) \in S\zeta_n^*(q)$, where L_a is the integral operator defined by (4.3.1).*

Chapter 5

Strong differential superordinations

The fifth chapter treats first order strong differential superordinations, best subordinant of the strong differential superordination, strong differential superordinations obtained with known operators.

5.1 First-order strong differential superordinations

In this section we study the special case of first order strong differential superordinations. The results are original and can be found in the paper [98].

In the paper [91], using the definitions given by Pommerenke [108], and Miller and Mocanu [76], the author introduced the notion of strong subordination (or Loewner) chain (see Definition 1.13.5) as follows:

Definition 5.1.1 [91] The function $L : U \times \overline{U} \times [0,\infty) \to \mathbb{C}$ is a strong subordination (or a Loewner) chain if $L(z,\zeta;t)$ is analytic and univalent in U for $\zeta \in \overline{U}$, $t \geq 0$, $L(z,\zeta;t)$ is continuously differentiable function of t on $[0,\infty)$ for all $z \in U$, $\zeta \in \overline{U}$, and $L(z,\zeta;s) \prec\prec L(z,\zeta;t)$ where $0 \leq s \leq t$.

Theorem 5.1.1 [98] *Let $h_1(z,\zeta)$ be convex in U, for all $\zeta \in \overline{U}$ with $h_1(0,\zeta) = a$, $\gamma \neq 0$ with* $\operatorname{Re}\gamma > 0$ *and $p \in \mathcal{H}\zeta[a,1] \cap Q$. If $p(z,\zeta) + \dfrac{zp'(z,\zeta)}{\gamma}$ is univalent in U, for all $\zeta \in \overline{U}$, and*

$$h_1(z,\zeta) \prec\prec p(z,\zeta) + \frac{zp'(z,\zeta)}{\gamma}, \tag{5.1.1}$$

$$q_1(z,\zeta) = \frac{\gamma}{z^\gamma}\int_0^z h_1(t,\zeta)t^{\gamma-1}dt, \tag{5.1.2}$$

then

$$q_1(z,\zeta) \prec\prec p(z,\zeta).$$

The function $q_1(z,\zeta)$ is convex and is the best subordinant.

Theorem 5.1.2 [98] *Let $q(z,\zeta)$ be convex in U, for all $\zeta \in \overline{U}$ and let $h(z,\zeta)$ be defined by*

$$q(z,\zeta) + \frac{zq'(z,\zeta)}{\gamma} = h(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \operatorname{Re}\gamma > 0. \tag{5.1.3}$$

If $p(z,\zeta) \in \mathcal{H}\zeta[a,1] \cap Q$*,* $p(z,\zeta) + \dfrac{zp'(z,\zeta)}{\gamma}$ *is univalent in* U*, for all* $\zeta \in \overline{U}$ *and satisfy*

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{zp(z,\zeta)}{\gamma}, \tag{5.1.4}$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

where

$$q(z,\zeta) = \frac{\gamma}{z^\gamma}\int_0^z h(t,\zeta)t^{\gamma-1}dt.$$

The function q *is the best subordinant.*

Remark 5.1.1 This last theorem is an example of solution to Problem 3 referred to in Chapter I, section 1.20.

The remaining theorem is an example of solution to Problem 2. It involve differential superordinations for which the subordinate function h is a starlike function.

Theorem 5.1.3 [98] *Let* $h(z,\zeta)$ *be starlike in* U*, for all* $\zeta \in \overline{U}$*, with* $h(0,\zeta) = 0$*. If* $p(z,\zeta) \in \mathcal{H}\zeta[0,1] \cap Q$ *and* $zp'(z,\zeta)$ *is univalent in* U*, for all* $\zeta \in \overline{U}$*, then*

$$h(z,\zeta) \prec\prec zp(z,\zeta) \tag{5.1.5}$$

implies

$$q(z,\zeta) \prec\prec p(z,\zeta),$$

where

$$q(z,\zeta) = \int_0^z h(t,\zeta)t^{\gamma-1}dt. \tag{5.1.6}$$

The function $q(z,\zeta) \in K\zeta$ *is the best subordinant.*

5.2 Best subordinant of the strong differential superordination

In this section we study best subordinant of certain strong differential superordination. The results are original and can be found in the paper [99].

For Ω_ζ a domain in $\mathbb{C}$, we now consider the strong differential superordination relation as follows:

$$\Omega_\zeta \subset \{\varphi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta)\}, \quad z \in U, \zeta \in \overline{U}. \tag{5.2.1}$$

Theorem 5.2.1 [99] *Let* $\Omega_\zeta \in \mathbb{C}$*, fie* $q(\cdot,\zeta) \in \mathcal{H}\zeta[a,n]$ *and let* $\psi \in \Psi_n[\Omega_\zeta, q(\cdot,\zeta)]$*. If* $p(\cdot,\zeta) \in Q(a)$ *şi* $\psi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta)$ *is univalent in* U *for all* $\zeta \in \overline{U}$*, then*

$$\Omega_\zeta \subset \{\psi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta)\}, \tag{5.2.2}$$

implies

$$q(z,\zeta) \prec\prec p(z,\zeta).$$

We next consider the special situation when $h(z,\zeta)$ is analytic on $U\times\overline{U}$ and $h(U\times\overline{U})=\Omega_\zeta\neq\mathbb{C}$, then the Theorem 5.2.1 becomes:

Theorem 5.2.2 [99] *Let $q(z,\zeta)\in\mathcal{H}\zeta[a,n]$, $h(z,\zeta)$ be analytic in $U\times\overline{U}$ and let $\psi\in\Psi_n[h(z,\zeta),q(z,\zeta)]$. If $p(z,\zeta)\in Q(a)$ and $\psi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$ is univalent in U for all $\zeta\in\overline{U}$, then*

$$h(z,\zeta)\prec\prec\psi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$$

implies

$$q(z,\zeta)\prec\prec p(z,\zeta).$$

Remark 5.2.1 The conclusion of the Theorem 5.2.2 can be written in the generalized form:

$$h(w(z),\zeta)\prec\prec\psi(p(w(z)),w(z)p'(w(z),\zeta),w^2(z)p''(w(z),\zeta);w(z),\zeta)$$

where $w:U\to U$, $z\in U$, $\zeta\in\overline{U}$.

The result from Theorem 5.2.2 can be extended to those cases in which the behavior of $q(z,\zeta)$ on the boundary of U is unknown, by the following theorem.

Theorem 5.2.3 [99] *Let $h(z,\zeta)$ and $q(z,\zeta)$ be univalent in U for all $\zeta\in\overline{U}$, with $q(0,\zeta)=a$ and set $q_\rho(z,\zeta)=q(\rho z,\zeta)$ and $h_\rho(z,\zeta)=h(\rho z,\zeta)$. Let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$ satisfy one of*

(i) $\varphi\in\phi_n[h(z,\zeta),q_\rho(z,\zeta)]$, for some $\rho\in(0,1)$, or

(ii) there exists $\rho_0\in(0,1)$ such that $\varphi\in\phi_n[h_\rho(z,\zeta),q_\rho(z,\zeta)]$, for all $\rho\in(\rho_0,1)$.

If $p(z,\zeta)\in\mathcal{H}\zeta[a,n]$, $\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$ is univalent in U for all $\zeta\in\overline{U}$ and

$$h(z,\zeta)\prec\prec\psi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta) \tag{5.2.3}$$

then

$$q(z,\zeta)\prec\prec p(z,\zeta).$$

The following theorem provides the existence of the best subordinant of (5.2.3) for certain φ and also provide a method for finding the best subordinant for the cases $n=1$ and $n>1$.

Theorem 5.2.4 [99] *Let $h(z,\zeta)$ be univalent in U for all $\zeta\in\overline{U}$ and let $\varphi:\mathbb{C}^3\times U\times\overline{U}\to\mathbb{C}$. Suppose that the differential equation*

$$\psi(q(z,\zeta),zq'(z,\zeta),z^2q''(z,\zeta);z,\zeta)=h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U} \tag{5.2.4}$$

has a solution $q(z,\zeta)\in Q(a)$. If $\varphi\in\phi[h(z,\zeta),q(z,\zeta)]$, $p(z,\zeta)\in Q(a)$ and $\varphi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta)$ is univalent in U for all $\zeta\in\overline{U}$, then

$$h(z,\zeta)\prec\prec\psi(p(z,\zeta),zp'(z,\zeta),z^2p''(z,\zeta);z,\zeta) \tag{5.2.5}$$

implies

$$q(z,\zeta)\prec\prec p(z,\zeta)$$

and $q(z,\zeta)$ is the best subordinant.

From this theorem we see that problem of finding the best subordinant of (5.2.5) essentially reduces to showing that differential equation (5.2.4) has a univalent solution and checking that $\varphi \in \phi[h(z,\zeta), q(z,\zeta)]$. The conclusion of the theorem can be written in the symmetric form

$$\psi(q(z,\zeta), zq'(z,\zeta), z^2q''(z,\zeta); z,\zeta) \prec\prec \psi(p(z,\zeta), zp'(z,\zeta), z^2p''(z,\zeta); z,\zeta) \tag{5.2.6}$$

implies

$$q(z,\zeta) \prec\prec p(z,\zeta).$$

5.3 Strong differential superordinations obtained by differential operators

In this section we obtain new Strong differential superordinations using Ruscheweyh derivative and Sălăgean differential operator. The results are original and are found in the papers [132] and [133].

Theorem 5.3.1 [132] *Let $q(z,\zeta)$ be in class $K\zeta$ with $q(0,\zeta) = 1$, and $h(z,\zeta)$ be defined by*

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{m+1} zq'(z,\zeta), \quad z \in U, \zeta \in \overline{U}. \tag{5.3.1}$$

Let $f \in A\zeta$ and we assume that $[R^{m+1}f(z,\zeta)]'$ is an univalent function and $[R^m f(z,\zeta)]' \in \mathcal{H}\zeta[1,1] \cap Q$.
If the strong differential superordination

$$h(z,\zeta) \prec\prec [R^{m+1}f(z,\zeta)]', \tag{5.3.2}$$

holds, then

$$q(z,\zeta) \prec\prec [R^m f(z,\zeta)]'$$

and this is the best result.

Theorem 5.3.2 [132] **Theorem 2.2.** *Let $h(z,\zeta)$ an analytic function from $U \times \overline{U}$, with $h(0,\zeta) = 1$, $h'(0,\zeta) \neq 0$, $z \in U, \zeta \in \overline{U}$, that satisfies the inequality*

$$Re[1 + \frac{zh''(z,\zeta)}{h'(z,\zeta)}] > -\frac{1}{2(m+1)}, m \geq 0. \tag{5.3.3}$$

Let $f \in A\zeta$ and we assume that $[R^{m+1}f(z,\zeta)]'$ is an univalent function and $[R^m f(z,\zeta)]' \in \mathcal{H}\zeta[1,1] \cap Q$.
If the strong differential superordination

$$h(z,\zeta) \prec\prec [R^{m+1}f(z,\zeta)]', \tag{5.3.4}$$

holds, then

$$q(z,\zeta) \prec\prec [R^m f(z,\zeta)]'$$

where

$$q(z,\zeta) = \frac{m+1}{z^{m+1}} \int_0^z h(t,\zeta) t^m dt. \tag{5.3.5}$$

The function $q(z,\zeta) \in K\zeta$ and is the best subordinant.

Theorem 5.3.3 [133] *Let $h(z,\zeta)$ be an analytic function from $U\times\overline{U}$, with $h(0,\zeta)=1$, $h'(0,\zeta)\neq 0$, $z\in U,\zeta\in\overline{U}$, that satisfies the inequality*

$$Re[1+\frac{zh''(z,\zeta)}{h'(z,\zeta)}]>-\frac{1}{2},\quad z\in U,\zeta\in\overline{U}.$$

Let $f\in A\zeta$ and we assume that $[S^{m+1}f(z,\zeta)]'$ is an univalent function and $[S^m f(z,\zeta)]'\in\mathcal{H}\zeta[1,1]\cap Q$.
If the strong differential superordination

$$h(z,\zeta)\prec\prec[I^{m+1}f(z,\zeta)]', \tag{5.3.6}$$

holds, then

$$q(z,\zeta)\prec\prec[I^m f(z,\zeta)]'$$

where

$$q(z,\zeta)=\frac{1}{z}\int_0^z h(t,\zeta)dt.$$

The function $q(z,\zeta)\in K\zeta$ and is the best subordinant.

Theorem 5.3.4 [133] *Let $q(z,\zeta)\in K\zeta$ and $h(z,\zeta)$ be defined by*

$$h(z,\zeta)=q(z,\zeta)+zq'(z,\zeta),\quad z\in U,\zeta\in\overline{U}. \tag{5.3.7}$$

Let $f\in A\zeta$ and we assume that $[S^m f(z,\zeta)]'$ is an univalent function and $\frac{S^m f(z,\zeta)}{z}\in\mathcal{H}\zeta[1,1]\cap Q$.
If the strong differential superordination

$$h(z,\zeta)\prec\prec[I^m f(z,\zeta)]', \tag{5.3.8}$$

holds, then

$$q(z,\zeta)\prec\prec\frac{I^m f(z,\zeta)}{z}$$

where q is given by (5.3.5).
The function q is the best subordinant.

Chapter 6

Harmonic functions

This chapter, dedicated to the harmonic functions, is structured in five sections. The first four contain basic notions for harmonic functions and harmonic mappings, treated the canonic representation of an harmonic function and the class S_H^0 of harmonic univalent functions. The last section contains original results, we define and investigating a new class of harmonic multivalent functions defined in the open unit disc, under certain conditions involving a new generalized differential operator.

The results of the sections $6.1 - 6.4$ can be found in the well- known papers P. Duren [31], P. Hamburg, P. T. Mocanu, N. Negoescu [51], James Clunie and Terry Sheil-Small [28].

6.1 Basic notions for harmonic functions. Harmonic mappings

Definition 6.1.1 A real function $u(x, y)$, $u : D \subset \mathbb{R}^2 \to \mathbb{R}$ is called harmonic function if it is satisfied Laplace equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Definition 6.1.2 A bijective transformation $u = u(x, y)$, $v = v(x, y)$ from a region D of the plain xOy into a region Ω of the plain uOv is a harmonic mapping if bouth u, v are harmonics.

Remark 6.1.1 It is convenient to use the complex notation

$$z = x + iy, \quad w = u + iv$$

with

$$w = f(z) = u(z) + iv(z).$$

Remark 6.1.2 From the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and from the existence of superior derivatives, it goes that each analytic function is an harmonic function.

Definition 6.1.3 The Jacobian of the function $f = u + iv$ is

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

If f is an analytic function, then its Jacobian has the following form:

$$J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2.$$

For analytic functions f, $J_f(z) \neq 0$ if and only if f is local univalent in z. Hans Lewy has showed in 1936 that this statement remain true for harmonic mappings.

By Lewy's theorem perspective, harmonic mappings are those which sense preserving with $J_f(z) > 0$, or those which sense reversing with $J_f(z) < 0$ in all domain D where f is univalent.

6.2 Canonic representation of an harmonic function

In a simple conex domain $D \subset \mathbb{C}$, a complex harmonic functions has the canonic representation $f = h + \overline{g}$, where h and g are analytic functions in D.

Remark 6.2.1 The function h is the analytic part of f and the function $\overline{g}$ is the coanalytic part of f.

Remark 6.2.2 In any simple conex domain we can write $f = h + \bar{g}$, where h and g analytic in D. A necessary and sufficient condition for f to be multivalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

6.3 The class S_H^0 of harmonic univalent functions

A harmonic function $f = h + \overline{g}$ from the open unit disc U can be expressed as

$$f(re^{i\theta}) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta}, \quad 0 \leq r < 1,$$

where

$$h(z) = \sum_{0}^{\infty} a_n z^n, \quad g(z) = \sum_{1}^{\infty} \overline{a}_n z^n.$$

Definition 6.3.1 We note with S_H the class of all harmonic sense preserving mappings, defined in the open unit disc U, normalized and univalent.
Then a harmonic mapping f from S_H has the representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

are analytic functions in U, with $h(0) = 0$, $h'(0) = 1$, $a_0 = b_0 = 0$ and $a_1 = 1$.

Remark 6.3.1 Class S_H is normal.

Definition 6.3.2 The class of functions $f \in S_H$ with $g'(0) = 0$ is noted with S_H^0,

$$S_H^0 = \{f \in S_H : \ g'(0) = b_1 = 0\}.$$

Theorem 6.3.1 (Clunie şi Sheil Small) [28] *The class* S_H^0 *is compact and normal.*

6.4 Harmonic multivalent functions defined by derivative operator

In this section we define and investigating a new class of harmonic multivalent functions defined in the open unit disc, under certain conditions involving a new generalized differential operator. Coefficient inequalities, distortion bounds and a covering result are also obtained. Furthermore, a representation theorem, an integral property and convolution conditions for the subclass denoted by $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ are also obtained. Finally, we will give an application of neighborhood.

The results are original and are obtained through collaboration, and can be found in the papers [24], [25].

Denote by $S_{\mathcal{H}}(p, n)$, $(p, n \in \mathbb{N} = \{1, 2, \ldots\})$, the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc U for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_{\mathcal{H}}(p, n)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1. \tag{6.4.1}$$

Let $\tilde{S}_{\mathcal{H}}(p, n, m)$, $(p, n \in \mathbb{N}, m \in \mathbb{N}_0 \cup \{0\})$, denote the family of functions $f_m = h + \bar{g}_m$ that are harmonic in D with the normalization

$$h(z) = z^p - \sum_{k=p+n}^{\infty} |a_k| z^k, \quad g_m(z) = (-1)^m \sum_{k=p+n-1}^{\infty} |b_k| z^k, \quad |b_{p+n-1}| < 1. \tag{6.4.2}$$

***1.* Coefficient bounds for the new classes $AL_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ and $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$**

We propose for the beginning a new generalized differential operator as follows.

Definition 6.4.1 [24], [25] Let $H(U)$ denote the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ and let $\mathcal{A}(p)$ be the subclass of the functions belonging to $H(U)$ of the form

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k.$$

For $m \in \mathbb{N}_0$, $\lambda \geq 0$, $\delta \in \mathbb{N}_0$, $l \geq 0$ we define the generalized differential operator $I_{\lambda,\delta}^m(p, l)$ on $\mathcal{A}(p)$ by the following infinite series

$$I_{\lambda,\delta}^m(p, l) h(z) = (p + l)^m z^p + \sum_{k=p+n}^{\infty} [p + \lambda(k - p) + l]^m C(\delta, k) a_k z^k, \tag{6.4.3}$$

where

$$C(\delta, k) = \begin{pmatrix} k + \delta - 1 \\ \delta \end{pmatrix} = \frac{\Gamma(k + \delta)}{\Gamma(k)\Gamma(\delta + 1)}. \tag{6.4.4}$$

Definition 6.4.2 [24], [25] Let $f \in S_{\mathcal{H}}(p,n)$, $p \in \mathbb{N}$. Using the operator (**??**) for $f = h + \bar{g}$ given by (**??**) we define the differential operator of f as

$$I^m_{\lambda,\delta}(p,l)f(z) = I^m_{\lambda,\delta}(p,l)h(z) + (-1)^m \overline{I^m_{\lambda,\delta}(p,l)g(z)} \tag{6.4.5}$$

where

$$I^m_{\lambda,\delta}(p,l)h(z) = (p+l)^m z^p + \sum_{k=p+n}^{\infty} [p+\lambda(k-p)+l]^m C(\delta,k) a_k z^k \tag{6.4.6}$$

and

$$I^m_{\lambda,\delta}(p,l)g(z) = \sum_{k=p+n-1}^{\infty} [p+\lambda(k-p)+l]^m C(\delta,k) b_k z^k. \tag{6.4.7}$$

In the following definitions we introduce new classes of harmonic multivalent functions by means of the generalized differential operator (6.4.5).

Definition 6.4.3 [24], [25] A function $f \in S_{\mathcal{H}}(p,n)$ is said to be in the class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if

$$\frac{1}{p+l}\mathrm{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^m_{\lambda,\delta}(p,l)f(z)}\right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{6.4.8}$$

where $I^m_{\lambda,\delta}f$ is defined by (6.4.5), for $m \in \mathbb{N}_0$.

Finally, we define the subclass

$$\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \equiv AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \cap \tilde{S}_{\mathcal{H}}(p,n,m). \tag{6.4.9}$$

In this paper we will give sufficient conditions for functions $f = h + \bar{g}$, where h and g are given by (**??**), to be in the class $AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. These coefficient conditions are also shown to be also necessary for functions in the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. Furthermore, distortion bounds for the subclass $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ are also obtained.

We first prove sufficient conditions for functions to be in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, in the following theorem.

Theorem 6.4.1 [24], [25] *Let $f = h + \bar{g}$ be given by (6.4.1). If*

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| + \tag{6.4.10}$$

$$+\sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \leq 1,$$

cu $\lambda n \geq \alpha(p+l)$,
where

$$d_{p,k}(m,\lambda,l) = [p+\lambda(k-p)+l]^m \tag{6.4.11}$$

then $f \in AL_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 6.4.2 [24] *Let $f_m = h + \bar{g_m}$ be given by (??). Then $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if and only if*

$$(6.4.12)\qquad \sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| +$$

$$+\sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \leq 1,$$

where $\lambda n \geq \alpha(p+l)$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda \geq 0$ and $d_{p,k}(m,\lambda,l)$ is given in (6.4.11).

2. **Distortion bounds**

The following theorem gives the distortion bounds for functions in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ which yields a covering result for this class.

Theorem 6.4.3 [24] *Let $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, with $0 \leq \alpha < 1$, $\lambda n \geq \alpha(p+l)$, $m \in \mathbb{N}_0$, $\lambda \geq 0$. Then for $|z| = r < 1$ one obtains*

$$(6.4.13)\qquad |f(z)| \leq (1+|b_{p+n-1}|r^{n-1})r^p + \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}$$

and

$$|f(z)| \geq (1-|b_{p+n-1}|r^{n-1})r^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda n]d_{p,n+p}(m,\lambda,l)C(\delta,n+p)} \cdot$$

$$\cdot \left\{1 - \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,n+p-1}(m,\lambda,l)C(\delta,n+p-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|\right\} r^{n+p}.$$

3. **Convex combination and extreme points**

In this section, we show that the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination of its members.

For $i = 1,2,3,...$, let the functions $f_{m_i}(z)$ be

$$(6.4.14)\qquad f_{m_i}(z) = z^p - \sum_{k=p+n}^{\infty} |a_{k,i}|z^k + (-1)^m \sum_{k=p+n-1}^{\infty} |b_{k,i}|\bar{z}^k.$$

Theorem 6.4.4 [25] *The class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination.*

Further, we will determine a representation theorem for functions in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ from which we also establish the extreme points of closed convex hulls of $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ denoted by $clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 6.4.5 [25] *Let $f_m(z)$ given by (??). Then $f_m(z) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if and only if*

$$(6.4.15)\qquad f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z),$$

where $h_p(z) = z^p$,

(6.4.16) $$h_k(z) = z^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}z^k,$$
$$k = p+n, p+n+1, ...,$$

and

(6.4.17) $$g_{m_k}(z) = z^p + (-1)^m \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\bar{z}^k,$$
$$k = p+n-1, p+n, ...,$$

cu $X_k \geq 0, Y_k \geq 0, X_p = 1 - \sum_{k=p+n}^{\infty} X_k - \sum_{k=p+n-1}^{\infty} Y_k$.

In particular, the extreme points of $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ *are* $\{h_k\}$ *and* $\{g_{m_k}\}$.

***4.* Integral property and convolution conditions**

In this section we will examine the closure properties of the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ under the generalized Bernardi-Libera-Livingston integral operator and also convolution properties of the same class.

Now, for $f = h + \bar{g}$ given by (6.4.1), we define the modified generalized Bernardi-Libera-Livingston integral operator of f as

(6.4.18) $$\mathcal{L}_c(f(z)) = \mathcal{L}_c(h(z)) + \overline{\mathcal{L}_c(g(z))}, \quad c > -p,$$

where
$$\mathcal{L}_c(h(z)) = \frac{c+p}{z^c}\int_0^z t^{c-1}h(t)dt$$
and
$$\mathcal{L}_c(g(z)) = \frac{c+p}{z^c}\int_0^z t^{c-1}g(t)dt.$$

Putting $g = 0$ in (6.4.18), we get the definition of the generalized Bernardi-Libera-Livingston integral operator on analytic functions, (see also [61], [62]).

Theorem 6.4.6 [25] *Let* $f \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. *Then* $\mathcal{L}_c(f)$ *belongs to the class* $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

For the harmonic functions

(6.4.19) $$f_1(z) = z^p - \sum_{k=p+n}^{\infty} |a_k|z^k + (-1)^m \sum_{k=p+n-1}^{\infty} |b_k|\bar{z}^k, \ |b_{p+n-1}| < 1,$$

and

(6.4.20) $$f_2(z) = z^p - \sum_{k=p+n}^{\infty} |A_k|z^k + (-1)^m \sum_{k=p+n-1}^{\infty} |B_k|\bar{z}^k, \ |B_{p+n-1}| < 1,$$

we define the convolution of f_1 and f_2 as
$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = z^p - \sum_{k=p+n}^{\infty} |a_k A_k|z^k + (-1)^m \sum_{k=p+n-1}^{\infty} |b_k B_k|\bar{z}^k.$$

In the following theorem, we examine the convolution properties of the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 6.4.7 [25] *For* $0 \leq \beta \leq \alpha < 1$ *let* $f_1 \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ *and* $f_2 \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. *Then* $f_1 * f_2 \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \subset \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$.

***5.* An application of neighborhood**

Let us define a generalized (n,η)-neighborhood of a function f given in (6.4.2), to be the set

$$N_{n,\eta}(f) = \Big\{ F_m(z) \in \tilde{S}_{\mathcal{H}}(p,n,m) :$$

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k - A_k| +$$

$$+ \sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k - B_k| \leq \eta \Big\}$$

where $F_m(z) = z^p - \sum_{k=p+n}^{\infty}|A_k|z^k + (-1)^m \sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k$.

Theorem 6.4.8 [25] *Let* $f_m = h + \bar{g}_m$ *be given by (6.4.2). If the functions* f_m *satisfy the conditions*

$$\sum_{k=p+n}^{\infty} k \cdot \Big[\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| + \tag{6.4.21}$$

$$+ \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \Big] \leq 1 - U_{p,\delta}^{\alpha}(m,\lambda,l)$$

and

$$\eta \leq \frac{p+n-\alpha-1}{p+n-\alpha}\left(1 - U_{p,\delta}^{\alpha}(m,\lambda,l)\right), \tag{6.4.22}$$

with $\lambda n \geq \alpha(p+l)$, *where*

$$U_{p,\delta}^{\alpha}(m,\lambda,l) = \frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,p+n-1}(m,\lambda,l)C(\delta,p+n-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|$$

then $N_{n,\eta}(f) \subset \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

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